Backpaper exam - December 26, 2023
B. Math. (Hons.) 2nd year
Group Theory
Instructor : B. Sury
Each question carries 9 points.

Q 1.
(i) Prove that a group is finite if, and only if, it has only finitely many different subgroups.
(ii) Let p be a fixed prime number. Consider the infinite group G consisting of all p-power roots of unity. Prove that each proper subgroup of G must be finite and cyclic.

Q 2.
(i) If G is a finite group, and H is a proper subgroup, prove that $G \neq \bigcup_{x \in G} Hx^{-1}$.
(ii) If $G = GL_2(\mathbb{C})$ and H is the subgroup of G that consists of the matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, prove that $G = \bigcup_{x \in G} Hx^{-1}$.
Hint for (i). If $xH = yH$, then $xHx^{-1} = yHy^{-1}$.

Q 3.
(i) Let G be a finite, nonabelian group generated by two elements x, y of order 2. Prove that G is isomorphic to a dihedral group.
(ii) Consider the functions $f(x) = -x$ and $g(x) = x + 1$. Under composition of functions, consider the group $\mathrm{Aff}(\mathbb{Z})$ of functions generated by the two functions f and g. Prove that $\mathrm{Aff}(\mathbb{Z})$ is an infinite, nonabelian group generated by two elements of order 2.
Hint for both parts. Consider the order of the product element xy in (i) and $f \circ g$ in (ii).

Q 4.
(i) If an abelian group A is isomorphic to the direct product $\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_r}$ where d_i's are positive integers satisfying $d_1|d_2|\cdots|d_r$, then the d_i's are uniquely determined by G.
(ii) Prove that a group of order 1365 cannot be simple.
Hint for (i). Look at the sets $\{a \in A : d_i A = 0\}$.
Hint for (ii). Look at p-Sylow subgroups.

Q 5.
(i) If G is a finite, nilpotent group, and H is a proper subgroup, prove that $N_G(H) \neq H$.
(ii) Give an example of a non-nilpotent group (could be infinite) having a normal subgroup N such that both N and G/N are nilpotent.
Hint for (ii). Any example for G in (ii) would be solvable.